Reinforcement Learning for Stochastic Networks

Exploiting Exponential Families to Tackle Nonconvexity

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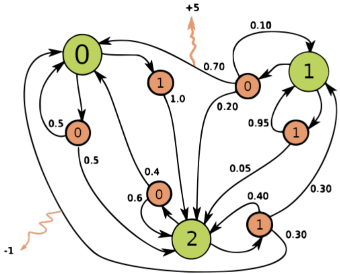
Workshop on restless bandits, index policies
and applications in reinforcement learning

Reinforcement learning

- **Markov decision process** (MDP)



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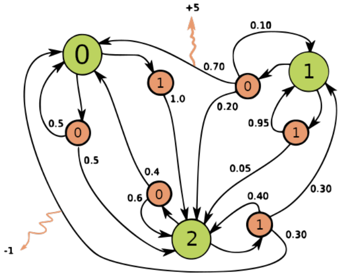

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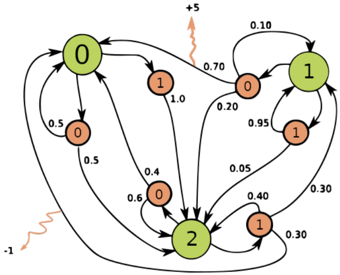

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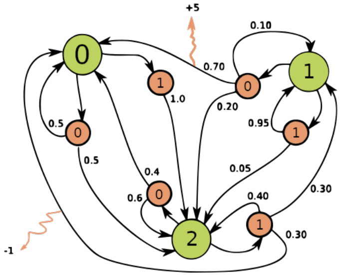

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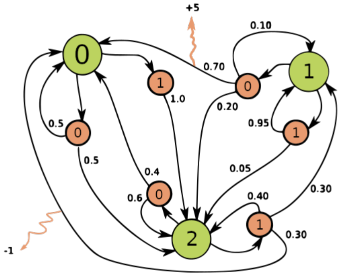

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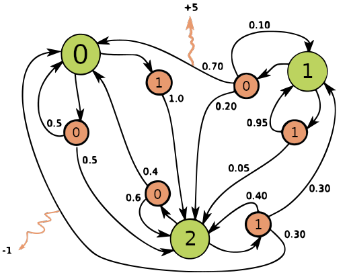

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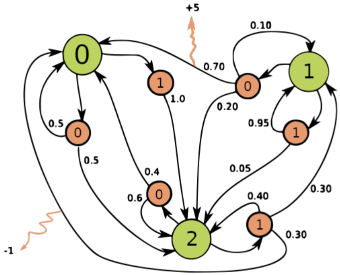

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Policy-gradient algorithms

- Typical **policy-gradient algorithm**:

```
Initialize $S_0$ and $\Theta_0$
for $t = 0, 1, 2, \ldots$ do
    Sample $A_t \sim \pi(\cdot|S_t, \Theta_t)$
    Take action $A_t$ and observe $S_{t+1}, R_{t+1}$
    Estimate $[\![\nabla J(\Theta_t)]\!]$ using the history $S_0, \Theta_0, A_0, R_1, \ldots, S_t, \Theta_t, A_t, R_{t+1}, S_{t+1}$
    Update $\Theta_{t+1} \leftarrow \Theta_t + \alpha[\![\nabla J(\Theta_t)]\!]$
end for
```

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  2: **for** $t = 0, 1, 2, \ldots$ **do**
  3: Sample $A_t \sim \pi(\cdot | S_t, \Theta_t)$
  4: Take action A_t and observe S_{t+1}, R_{t+1}
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```
Initialize S_0 and Θ_0
for t = 0, 1, 2, ... do
    Sample A_t ~ π(·|S_t, Θ_t)
    Take action A_t and observe S_{t+1}, R_{t+1}
    Estimate [[∇J(Θ_t)]] using the history S_0, Θ_0, A_0, R_1, ..., S_t, Θ_t, A_t, R_{t+1}, S_{t+1}
    Update Θ_{t+1} ← Θ_t + α[[∇J(Θ_t)]]
end for
```

How?

- **Actor-critic** applies the policy-gradient theorem (Sutton and Barto, 2018):

$$[\![\nabla J(\Theta_t)]\!] \leftarrow (R_{t+1} - [\![\mathbb{E}[R]]\!] + [\![v(S_{t+1})]\!] - [\![v(S_t)]\!])\nabla \log \pi(A_t|S_t, \Theta_t).$$

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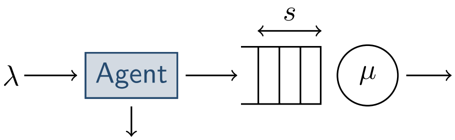
- Can we do better by exploiting the **system structure**?

Example: M/M/1 queue with admission control

- Arrival rate $\lambda > 0$, service rate $\mu > 0$
- State: queue length $s \in \{0, 1, 2, \dots\}$
- Actions: accept or reject
- Admission reward α per job
- Holding cost rate η per job per time unit



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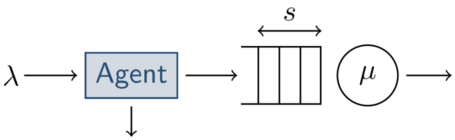

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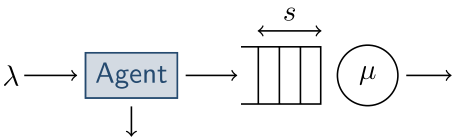

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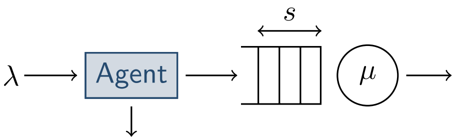

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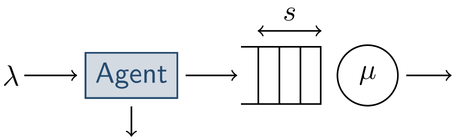

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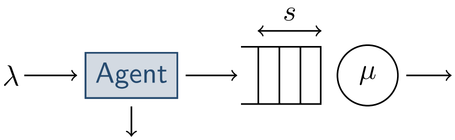

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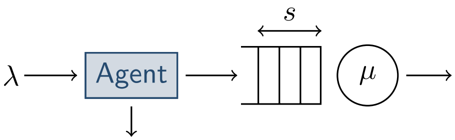

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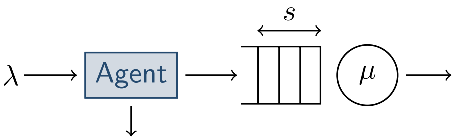

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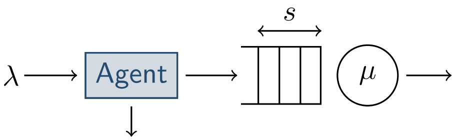

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- Main contributions:
  1. Product-form distributions as exponential families
  2. Score-aware gradient estimator (SAGE)
  3. SAGE-based policy-gradient algorithm
  4. Nonconvex convergence result

① Product-form distributions as exponential families

- **Product-form** distribution

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① Product-form distributions as exponential families

- **Product-form** distribution ⟶ **Exponential family** of distributions

$$p(s|\theta) = \frac{1}{Z(\theta)} \prod_{i=1}^{n} \rho_i(\theta)^{x_i(s)} \qquad \log p(s|\theta) = \log \rho(\theta)^\intercal x(s) - \log Z(\theta)$$

- **Feature function** $x = (x_1, x_2, \ldots, x_n)$ ⟶ **Feature function** $x = (x_1, x_2, \ldots, x_n)$
- **Load function** $\rho = (\rho_1, \rho_2, \ldots, \rho_n)$ ⟶ **Log-load function** $\log \rho = (\log \rho_1, \ldots, \log \rho_n)$
- **Partition function** Z ⟶ **Log-partition function** $\log Z$

$$Z(\theta) = \sum_s \prod_{i=1}^{n} \rho_i(\theta)^{x_i(s)} \qquad \log Z(\theta) = \log\left(\sum_s \mathrm{e}^{\log \rho(\theta)^\intercal x(s)}\right)$$

② Score-aware gradient estimator (SAGE)

- The **score** is the gradient of the log-likelihood with respect to the parameter vector:

$$\text{“Likelihood”} = p(s|\theta) \quad \to \quad \text{“Score”} = \nabla \log p(s|\theta).$$

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Theorem

Recalling that $(S, A, R) \sim$ *stationary distribution of* $((S_t, A_t, R_{t+1}), t \geq 0)$*, we have*

$$\begin{aligned} \nabla \log p(s|\theta) &= \mathrm{D} \log \rho(\theta)^\intercal (x(s) - \mathbb{E}[x(S)]), \\ \nabla J(\theta) &= \mathrm{D} \log \rho(\theta)^\intercal \mathrm{Cov}[R, x(S)] + \mathbb{E}[R \, \nabla \log \pi(A|S, \theta)]. \end{aligned}$$

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- **Main take-away:** If we can evaluate $\mathrm{D} \log \rho(\theta)$, this gives us an estimator for $\nabla J(\theta)$.

③ SAGE-based policy-gradient algorithm

- Typical **policy-gradient algorithm**:

Initialize S_0 and Θ_0
for $t = 0, 1, 2, \dots$ **do**
    Sample $A_t \sim \pi(\cdot | S_t, \Theta_t)$
    Take action A_t and observe S_{t+1}, R_{t+1}
    Estimate $[\![\nabla J(\Theta_t)]\!]$ using the history $S_0, \Theta_0, A_0, R_1, \dots, S_t, \Theta_t, A_t, R_{t+1}, S_{t+1}$ How?
    Update $\Theta_{t+1} \leftarrow \Theta_t + \alpha [\![\nabla J(\Theta_t)]\!]$
end for

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for t = 0, 1, 2, ... do
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    Take action A_t and observe S_{t+1}, R_{t+1}
    Estimate ⟦∇J(Θ_t)⟧ using the history S_0, Θ_0, A_0, R_1, ..., S_t, Θ_t, A_t, R_{t+1}, S_{t+1}    How?
    Update Θ_{t+1} ← Θ_t + α⟦∇J(Θ_t)⟧
end for
```

- **Actor-critic** applies the policy-gradient theorem (Sutton and Barto, 2018):

$$[\![\nabla J(\Theta_t)]\!] \leftarrow (R_{t+1} - [\![\mathbb{E}[R]]\!] + [\![v(S_{t+1})]\!] - [\![v(S_t)]\!])\nabla \log \pi(A_t|S_t, \Theta_t).$$

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  7: **end for**

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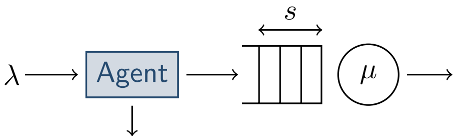
- We instead estimate $[\![\nabla J(\Theta_t)]\!]$ with a **score-aware gradient estimator (SAGE)**:

$$[\![\nabla J(\Theta_t)]\!] \leftarrow \mathrm{D} \log \rho(\Theta_t)^\intercal [\![\mathrm{Cov}[R, x(S)]]\!] + [\![\mathbb{E}[R \nabla \log \pi(A|S, \Theta_t)]]\!].$$

Example: M/M/1 queue with admission control

Stable case

- Arrival rate $\lambda = 0.7$, service rate $\mu = 1$
- Admission reward $\alpha = 5$
- Holding cost rate $\eta = 1$
- Initial policy $\pi(\Theta_0) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$
- Optimal policy $\pi_0(\theta^\star) = \pi_1(\theta^\star) = \pi_2(\theta^\star) = 1$ and $\pi_3(\theta^\star) = 0$.



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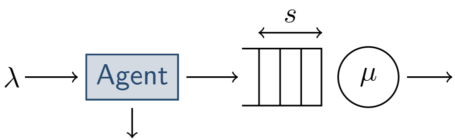

Possibly-unstable case

- Arrival rate $\lambda = 1.4$, service rate $\mu = 1$
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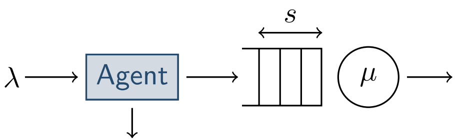

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Simulation setup

- 10^6 steps
- Convergence time T: $J(\Theta_t) > J(\theta^\star) - \epsilon$ for each $t \in \{T, T+1, \dots, 10^6\}$

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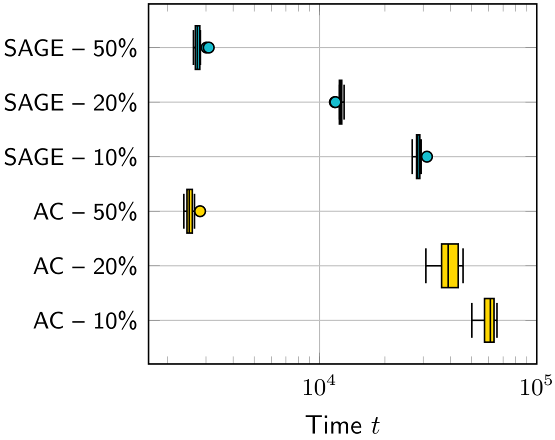
Stable case – Convergence times



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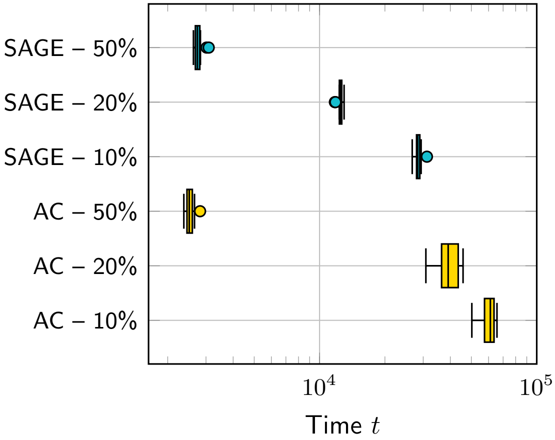

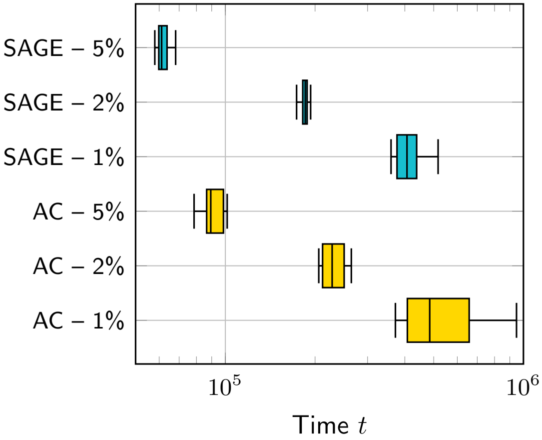

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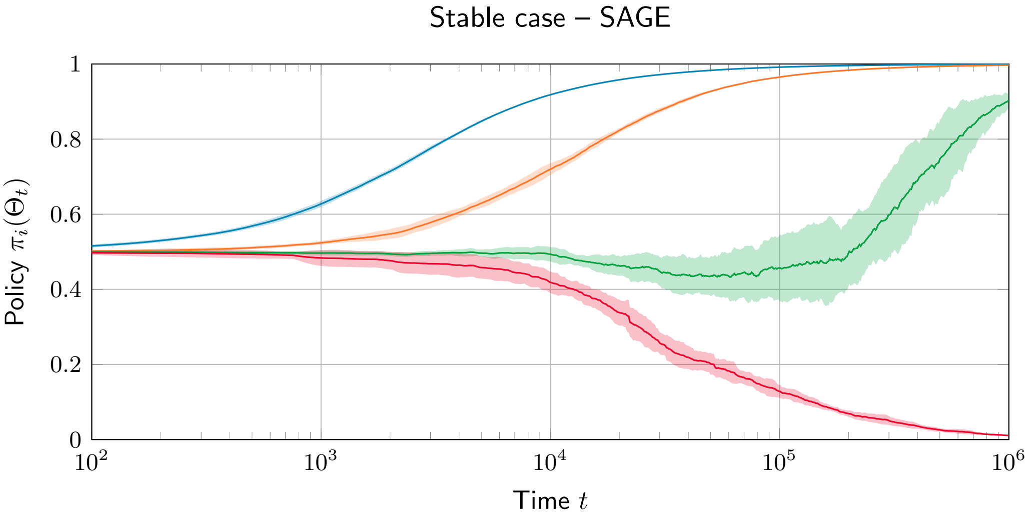

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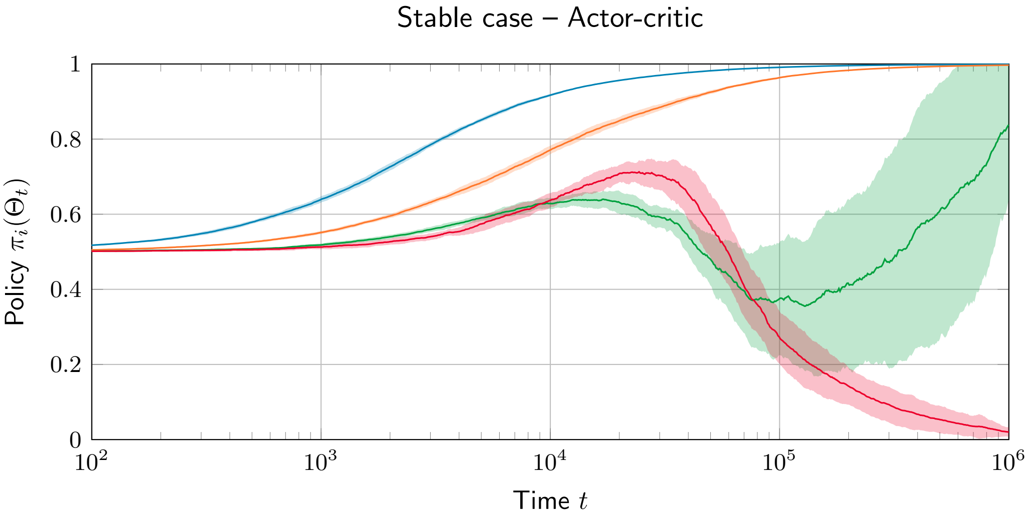

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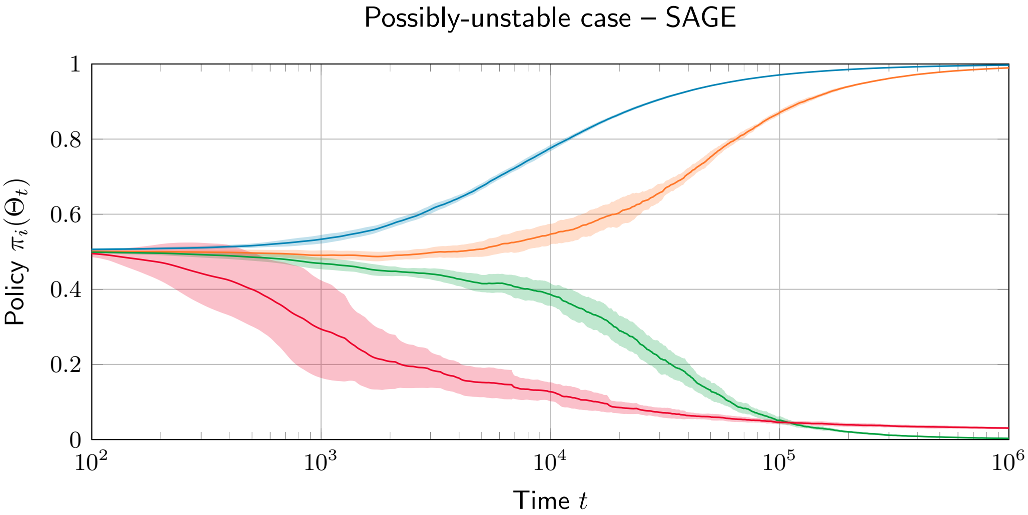

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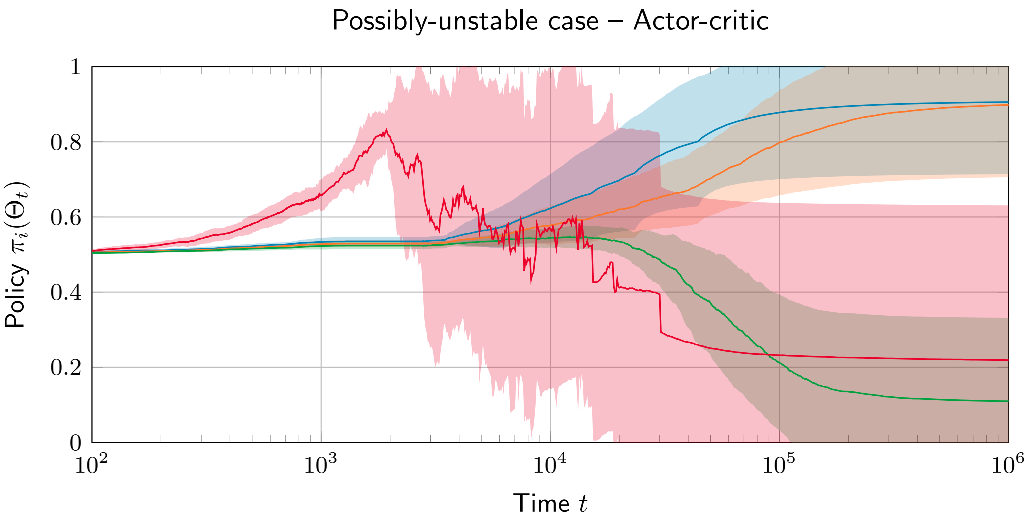

④ Local convergence result

Sketch of Theorem

Under additional assumptions, a batch variant of the algorithm that starts in a basin of attraction of a global maximizer will converge to a global maximizer with large probability.

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What are these "additional assumptions"?

- There exists a neighborhood of the global maximizer where:
  - The Markov chain of state-action pairs is geometrically ergodic.
  - The objective function behaves approximately in a convex manner in directions that are perpendicular to the set of global maximizers.
  - The function $\mathrm{D}\log\rho$ is bounded and the functions x, r, and $r\,\nabla\log\pi$ grow slowly enough.

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- The step sizes are decreasing and the batch sizes are increasing.

Conclusion

- **Main contributions**
  1. Product-form distributions as exponential families
  2. Score-aware gradient estimator (SAGE)
  3. SAGE-based policy-gradient algorithm
  4. Nonconvex convergence result

Product-form stationary distribution

$$\log p(s|\theta) = \log \rho(\theta)^\intercal x(s) - \log Z(\theta)$$

$$\downarrow$$

$$\nabla \log p(s|\theta) = \mathrm{D} \log \rho(\theta)^\intercal (x(s) - \mathbb{E}[x(S)])$$

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Score-aware gradient estimator (SAGE)

- **Future research directions**
  - Run extensive numerical results on more challenging examples.
  - Find better estimators for covariance and expectation, such as robust estimators.
  - Apply to (queueing) systems where the stationary distribution is known only *up to a multiplicative constant*.