# A unifying computation of Whittle's index for Markovian bandits

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Joint work with

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# Outline

# Background and overview

- A particular case of constrained Markov Decision Process (MDP).
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  - Machine repairman problem (Glazebrook et al., 2005), content delivery problem (Larrañaga et al., 2015)
  - Minimum job loss routing (Niño-Mora, 2012b), inventory routing (Archibald et al., 2009), processor sharing queues (Borkar and Pattathil, 2017), congestion control in TCP (Avrachenkov et al., 2013) etc.

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## Major challenges

- Establishing indexability and computations of Whittle's index.

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## Gittin's index

- For MABP, optimal policy is an index rule (Gittins et al., 2011).
- For example, $c\mu$ rule in multi-class queues.

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### Whittle's relaxation (Whittle, 1988)

Restriction on number of active bandits to be respected on *average* only.

- Optimal solution to the relaxed problem is of index type.
- The Whittle's index recovers Gittin's index for non-restless case.

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  - A generalization to several classes of bandits, arrivals of new bandits and multiple actions (Verloop, 2016).

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## Results

- A unifying framework for obtaining Whittle's index.
- Retrieve Whittle's indices in literature including machine repairman problem, content delivery problem etc.

# Model description and notations

$$
\begin{aligned}
K &: \text{Number of ongoing projects or bandits.} \\
a &: \text{Binary action to make the bandit active or passive.} \\
\phi &: \text{The policy to make a bandit active or passive.} \\
N_k^{\phi}(t) &: \text{State of bandit } k \text{ at time } t \text{ under policy } \phi. \\
S_k^{\phi}(\vec{N}^{\phi}(t)) \in \{0,1\} &: \text{Whether or not bandit } k \text{ is made active at time } t. \\
C_k(n,a) &: \text{Cost per unit of time when bandit } k \text{ is in state } n. \\
L_k^{\infty}(x,y,a) &: \text{The lump-sum cost for bandit } k \text{ when state instanteneously} \\
&\quad \text{changes form } x \text{ to } y \text{ under action } a.
\end{aligned}
$$

- Each bandit is modeled as continuous time Markov chain.
- Both *finite* and *infinite* transition rates are allowed.

# Objective

To minimize the long-run average cost:

$$\mathcal{C}^{\phi} := \limsup_{T\to\infty} \sum_{k=1}^{K} \frac{1}{T}\mathbb{E}\left(\int_0^T C_k(N_k^{\phi}(t), S_k^{\phi}(\vec{N}^{\phi}(t))) + C_k^{\infty,\phi}(N^{\phi}(t), S_k^{\phi}(\vec{N}^{\phi}(t)))dt\right) \tag{1}$$

The first term is a contribution from holding cost and the second term corresponds to the lump-sum cost due to impulses.

# Hard constraint

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  - Standard restless bandit constraint.
- If $f_k(N_k^{\phi}, S_k^{\phi}(\vec{N})) = N_k^{\phi} S_k^{\phi}(\vec{N})$, constraint (2) implies $\sum_{k=1}^{K} N_k^{\phi} S_k^{\phi}(\vec{N}) \leq M$.
  - Buffer constraint for TCP (Avrachenkov et al., 2013).
- $f_k(N_k^{\phi}, S_k^{\phi}(\vec{N}))$ represents the capacity occupation (volume) in state $N_k^{\phi}$ under action $S_k^{\phi}(\vec{N})$.
  - Family of sample path knapsack capacity allocation constraint (Jacko, 2016; Graczová and Jacko, 2014).

# Closed form expression for Whittle's index

**Theorem 1.**

*Assume an optimal solution of relaxed problem is of threshold type, and $\mathbb{E}(f_k(N_k^n, S_k^n(N_k^n)))$ is strictly increasing in n. Then, bandit k is indexable. If the structure of an optimal solution of relaxed problem is of 0-1 type, then, in case*

$$\frac{F_k^n(N_k^n, S_k^n(N_k^n)) - F_k^{n-1}(N_k^{n-1}, S_k^{n-1}(N_k^{n-1}))}{\mathbb{E}(f_k(N_k^n, S_k^n(N_k^n))) - \mathbb{E}(f_k(N_k^{n-1}, S_k^{n-1}(N_k^{n-1})))} \tag{3}$$

*is non-decreasing in n, Whittle's index $W_k(n_k)$ is given by (3) and is hence non-decreasing. Similarly, if the structure of an optimal solution of relaxed problem is of 1-0 type, then, in case (3) is non-decreasing in n, $-W_k(n_k)$ is given by (3) and hence Whittle's index is non-increasing.*

$F_k^n(N_k^n, S_k^n(N_k^n))$ is the expected cost under the threshold policy $n$ for bandit $k$.

# Finite transition rates

The transitions rates of vector $\vec{N} = (N_1, N_2, ..., N_K)$ are:

$$\begin{cases} \vec{N} \to \vec{N} + \vec{e}_k & \text{with transition rate } b_k^a(N_k) \\ \vec{N} \to \vec{N} - \vec{e}_k & \text{with transition rate } d_k^a(N_k) \\ \vec{N} \to \vec{N} + \alpha^a(n_k)\vec{e}_k & \text{with transition rate } h_k^a(N_k) \\ \vec{N} \to \vec{N} - \beta^a(n_k)\vec{e}_k & \text{with transition rate } l_k^a(N_k), \end{cases}$$

The long run average cost:

$$\mathcal{C}^\phi = \limsup_{T\to\infty} \sum_{k=1}^{K} \frac{1}{T}\mathbb{E}\left(\int_0^T C_k(N_k^\phi(t), S_k^\phi(\vec{N}^\phi(t)))dt\right) \tag{4}$$

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Machine repairman problem, class selection problem, load balancing problem.

# Examples of finite transition rates

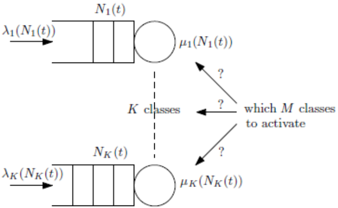


Figure: Class selection problem

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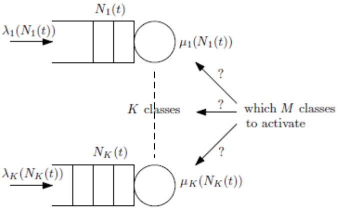


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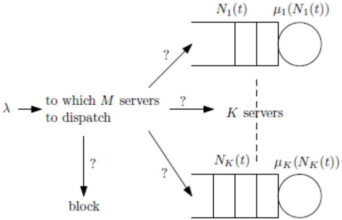


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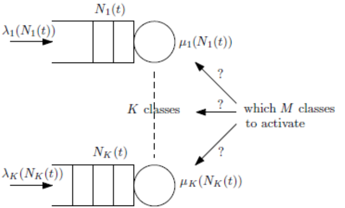


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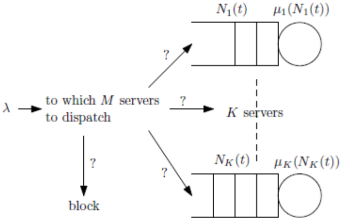


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- Machine repairman problem (Glazebrook et al., 2005)
  - $M$ machines to be repaired by $R$ repairmen.

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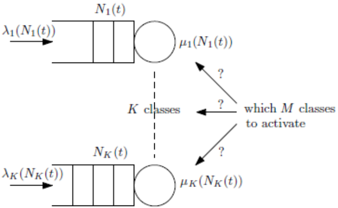


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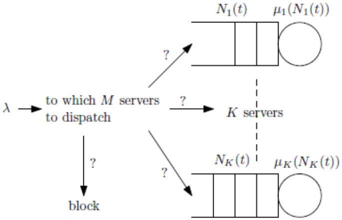


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- Machine repairman problem (Glazebrook et al., 2005)
  - $M$ machines to be repaired by $R$ repairmen.
- Load balancing problem (Argon et al., 2009)
  - With dedicated arrivals to each queues.

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The transition rates of vector $\vec{N} = (N_1, N_2, ..., N_K)$ for this case are:

$$\begin{cases} \vec{N} \to \vec{N} + \vec{e}_k & \text{with transition rate } b_k^a(N_k) \\ \vec{N} \to \vec{N} - \vec{e}_k & \text{with transition rate } d_k^a(N_k) \\ \vec{N} \to \vec{N} + \alpha^a(n_k)\vec{e}_k & \text{with transition rate } h_k^a(N_k) \\ \vec{N} \to \vec{N} - \beta^a(n_k)\vec{e}_k & \text{with transition rate } l_k^a(N_k) \\ \vec{N} \to \vec{N} + \gamma^a(n_k)\vec{e}_k & \text{with impulse rate } \tilde{h}_k^a(N_k), \\ \vec{N} \to \vec{N} - \delta^a(n_k)\vec{e}_k & \text{with impulse rate } \tilde{l}_k^a(N_k), \end{cases}$$

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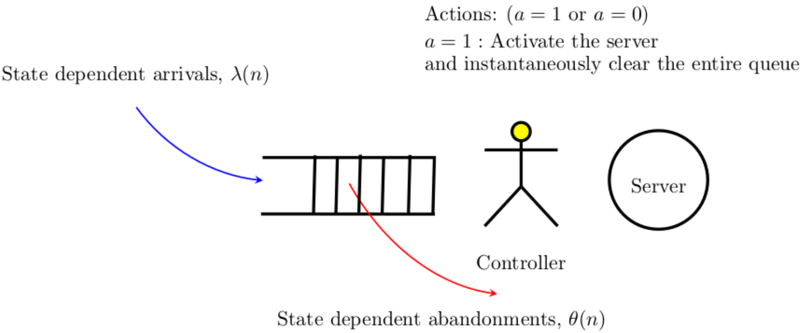
- Content delivery problem (Larrañaga et al., 2015).
- Instantaneous change in state.

# Content Delivery




State independent arrival and service rate (Larrañaga et al., 2015).

# Lagrangian Relaxation

$$\limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left(\int_0^T \sum_{k=1}^{K} f_k(N_k^{\phi}(t), S_k^{\phi}(\vec{N}^{\phi}(t)))dt\right) \leq M \text{ (On average)} \tag{5}$$

The unconstrained problem is to find a policy $\phi$ that minimizes

$$C^{\phi}(W) = \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left(\int_0^T \left(\sum_{k=1}^{K} C_k(N_k^{\phi}(t), S_k^{\phi}(\vec{N}^{\phi}(t)))\right.\right.$$

$$\left.\left. +C_k^{\infty,\phi}(N^{\phi}(t), S_k^{\phi}(\vec{N}^{\phi}(t))) - W\left(\sum_{k=1}^{K} f_k(N_k^{\phi}(t), S_k^{\phi}(\vec{N}^{\phi}(t))) - M\right) dt\right)\right), \tag{6}$$

# Decomposition

The problem can be decomposed (key observation in Whittle (1988)):

$$\mathbb{E}(C_k(N_k^{\phi}, S_k^{\phi}(N_k^{\phi}))) + C_k^{\infty,\phi}(N_k^{\phi}(t), S_k^{\phi}(\vec{N}^{\phi}(t))) - W\mathbb{E}(f_k(N_k^{\phi}, S_k^{\phi}(N_k^{\phi}))) \quad (7)$$

- The solution to the relaxed problem:
  - Combining the solution of $K$ separate problems.

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- The solution to the relaxed problem:
  - Combining the solution of $K$ separate problems.
- The decomposed problem is an MDP.
  - The optimal policy is the solution of the dynamic programming equations.
- Indexability and Whittle's index.

# Monotone policies

**Definition 1.**

*There is a threshold $n_k(W)$ such that when bandit $k$ is in a state $m_k \leq n_k(W)$, then action $a$ is optimal, and otherwise action $a'$ is optimal, $a,\ a' \in \{0, 1\}$ and $a \neq a'$.*

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- A policy $\phi = n$ denotes a threshold policy with threshold $n$,
  - 0-1 type if $a = 0$ and $a' = 1$
  - 1-0 type if $a = 1$ and $a' = 0$
- For certain problems, optimal solution of problem (7) is of threshold type.

# Optimality of threshold policies

**Proposition 1.**

*Consider the finite transition rates and assume*

$$\begin{aligned} b_k^a(N_k) &= \lambda_k^0(n_k)(1-a) \\ d_k^a(N_k) &= \mu_k^1(n_k)a + \mu_k^0(n_k)(1-a) \\ h_k^a(N_k) &= 0 \\ l_k^a(N_k) &= l_k^1(n_k)a + l_k^0(n_k)(1-a) \end{aligned}$$

*Then there exists an* $n_k \in \{-1, 0, 1, \ldots\}$ *such that a 0-1 type of threshold policy, with threshold* $n_k$*, optimally solves problem (7).*

Details of the proof

# 1-0 type policies

If instead,

$$\begin{aligned} b_k^a(N_k) &= \lambda_k^1(n_k)a + \lambda_k^0(n_k)(1-a) \\ d_k^a(N_k) &= \mu_k^0(n_k)(1-a) \\ h_k^a(N_k) &= h_k^1(n_k)a + h_k^0(n_k)(1-a) \\ l_k^a(N_k) &= 0 \end{aligned}$$

Then there exists an $n_k \in \{-1, 0, 1, ...\}$ such that a 1-0 type of threshold policy, with threshold $n_k$, optimally solves problem (7).

# Infinite transition rates

**Proposition 2.**

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l_k^a(N_k) &= l_k^0(n_k)(1-a) \\
\tilde{h}_k^a(N_k) &= 0 \\
\tilde{l}_k^a(N_k) &= \infty \text{ for } a = 1 \text{ (and 0 otherwise)}
\end{aligned}$$

*Then there exists an* $n_k \in \{-1, 0, 1, \ldots\}$ *such that a 0-1 type of threshold policy, with threshold* $n_k$*, optimally solves problem (7).*

## Applications

- Machine repairman problem
- Content delivery problem
- Load balancing problem

# Machine Repairman problem

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- Action $a = 1$ (use the repairman)
  - State improves.
  - Machine is returned to pristine state 0.
- Action $a = 0$
  - State further deteriorates.
  - Machine spends a random amount of time in its current damage state before deteriorating to the next one.

# Machine repairman problem

- Possibility of a catastrophic breakdown with rate $\psi_k(n_k)$
- Repair rates be $r_k(n_k)$ from state $n_k$.
- Deterioration rates be $\lambda_k(n_k)$.

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$$
\begin{aligned}
C_k^b(n_k, 0) &: \text{Huge lump cost for breakdown.} \\
C_k^r(n_k, 1) &: \text{Cost of using the repairman.} \\
C_k^{pd}(n_k, 0) &: \text{Per unit cost of deterioration.}
\end{aligned}
$$

$$C_k^b(n_k, 0) >> C_k^r(n_k, 1)$$

# Machine repairman problem

- Possibility of a catastrophic breakdown with rate $\psi_k(n_k)$
- Repair rates be $r_k(n_k)$ from state $n_k$.
- Deterioration rates be $\lambda_k(n_k)$.

$$C_k^b(n_k, 0) : \text{Huge lump cost for breakdown.}$$
$$C_k^r(n_k, 1) : \text{Cost of using the repairman.}$$
$$C_k^{pd}(n_k, 0) : \text{Per unit cost of deterioration.}$$

$$C_k^b(n_k, 0) >> C_k^r(n_k, 1)$$

**Objective**

To deploy the repairmen to minimize the average cost.

The Markov decision process is characterized by the following transition rates:

$$\begin{aligned} b_k^a(N_k) &= \lambda_k(n_k)(1-a) \\ d_k^a(N_k) &= 0 \\ h_k^a(N_k) &= 0 \\ l_k^a(N_k) &= r_k(n_k)a + \psi_k(n_k)(1-a) \end{aligned}$$

$$f_k(N_k^\phi, S_k^\phi(\vec{N})) = S_k^\phi(\vec{N})$$

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## Threshold optimality

0-1 type of threshold policy is optimal.

# Dynamics of a bandit in machine repairman problem

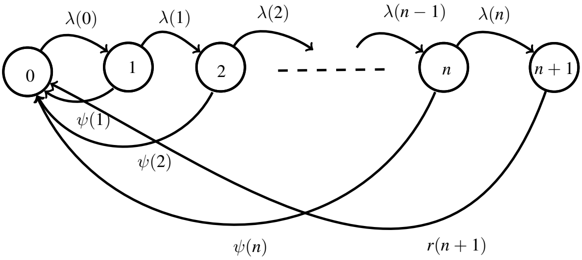


Figure: Transition diagram for threshold policy '$n$' of machine repairman problem

# Indexabiliity

**Lemma 1.**

*Machine k is indexable if repair rates are non-decreasing in its state, i.e., $r_k(n_k) \leq r_k(n_k + 1)\ \forall\ n_k$. In particular, all machines are indexable for state independent repair rates.*

- Follows from Theorem 1.
  - $\mathbb{E}(f_k(N_k^{n_k}, S_k^{n_k}(N_k^{n_k})))$ is strictly increasing in $n_k$.
- Equivalently, $\sum_{m=0}^{n_k} \pi_k^{n_k}(m)$ is strictly increasing in $n_k$ for machine repairman problem.

Details of the proof

# Whittle's index

## Lemma 2.

*The Whittle's index, $W_k(n)$, for machine $k$ is given by*

$$\frac{\left(C_{Sum}(n) + C_k^r(n+1,1)P_n\right)\left(P_{Sum}(n-1) + \frac{P_{n-1}}{r_k(n)}\right) - \left(C_{Sum}(n-1) + C_k^r(n,1)P_{n-1}\right)\left(P_{Sum}(n) + \frac{P_n}{r_k(n+1)}\right)}{\frac{P_{n-1}}{r_k(n)}\sum_{i=0}^{n}\frac{P_i}{\lambda_k(i)} - \frac{P_n}{r_k(n+1)}\sum_{i=0}^{n-1}\frac{P_i}{\lambda_k(i)}} \tag{8}$$

*where* $C_{Sum}(n) = \sum_{i=1}^{n}\left[(P_{i-1} - P_i)C_k^b(i,0) + \frac{P_i C_k^{pd}(i,0)}{\lambda_k(i)}\right]$, $P_{Sum}(n) = \sum_{i=0}^{n}\frac{P_i}{\lambda_k(i)}$,
$P_i = \prod_{j=1}^{i} p_k(j)$, $p_k(j) = \frac{\lambda_k(j)}{\lambda_k(j)+\psi_k(j)}$ *and* $P_0 = 1$, *if (8) is non-decreasing in $n$.*

- Follows from Theorem 1.

Details of the proof

## Known results

- For $1/r_k = 1$, we recover the index for average cost criterion in discrete time (see Equation (19) in Glazebrook et al. (2005)).

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- For $1/r_k = 1$, $C_k^{pr} = 0$ and $C_k^{pd}(n, 0) = C_k n$, we get the index which is consistent with the result of Whittle's approximate evaluation (See ch. 14.6 in Whittle (1996)).

# Dynamics of a bandit in content delivery problem

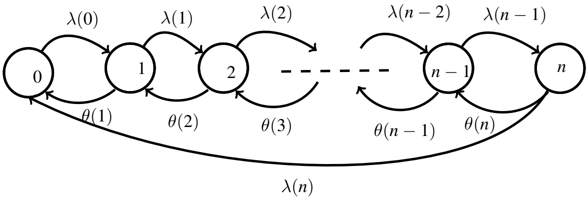


Figure: Transition diagram for threshold policy '$n$' in content delivery network

# Dynamics of a bandit in content delivery problem

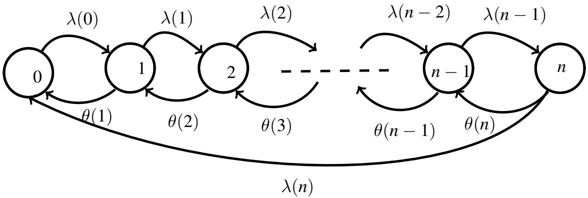


Figure: Transition diagram for threshold policy '$n$' in content delivery network

## Indexability

- Follows from Theorem 1.
- $\pi^n(n)$ is strictly decreasing in $n$.
- Index for state dependent cost and rates.

# Whittle's index

**Corollary 1.**

*If the rates and costs are state independent, i.e.,* $\lambda(i) = \lambda$, $\theta(i) = i\theta$, $C^h(i) = C^h$, $C^a(i) = C^a$ *and* $C_s^\infty(i) = C_s^\infty \ \forall$ *i, then, the Whittle's index is given by*

$$W(n) = \tilde{C}\frac{\mathbb{E}(N^n) - \mathbb{E}(N^{n-1})}{\pi^{n-1}(n-1) - \pi^n(n)} - \lambda C_s^\infty \tag{9}$$

*if (9) is non-decreasing in n, where* $\tilde{C} = C^h + \theta C^a$ *and* $\mathbb{E}(N^n)$ *is the expected number of jobs under threshold policy n.*

# Whittle's index

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*if (9) is non-decreasing in n, where $\tilde{C} = C^h + \theta C^a$ and $\mathbb{E}(N^n)$ is the expected number of jobs under threshold policy n.*

- Obtain the results in Larrañaga et al. (2015).

# Limited processor sharing (LPS-$d$)

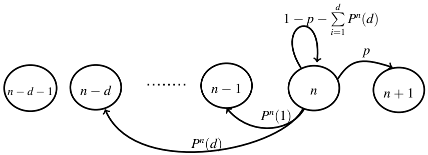



Figure: One step evolution of Markov chain in LPS-$d$ scheduling scheme.

- $d = 1$ implies FCFS (Argon et al., 2009).
- $d = \infty$ implies processor sharing (Borkar and Pattathil, 2017).

# Performance of the index policy

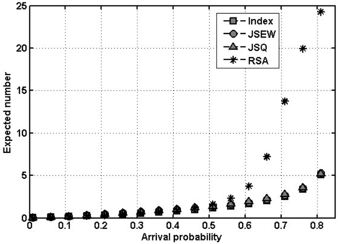



Figure: Index policy, JSQ, JSEW and RSA

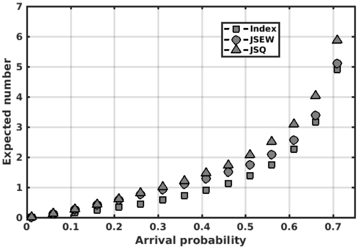
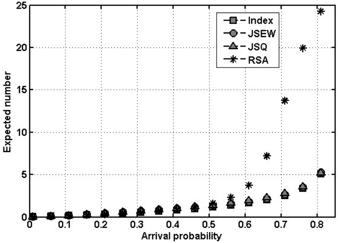



Figure: Index policy, JSQ and JSEW.

# Performance of the index policy

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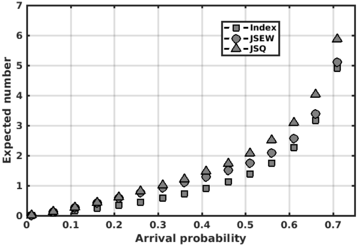



Figure: Index policy, JSQ and JSEW.

Index policy uniformly performs better.

# Performance of the index policy

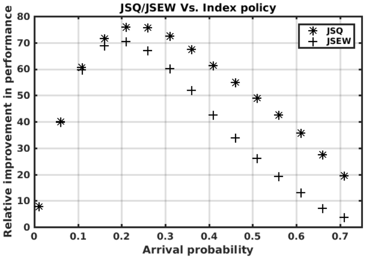



Figure: Percentage relative improvement.

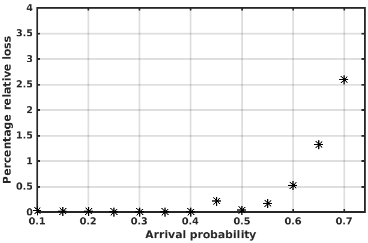
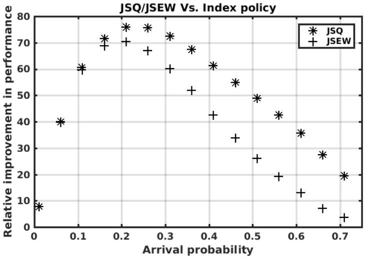



Figure: Comparision with the optimal policy.

# Performance of the index policy

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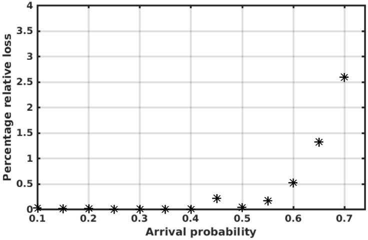


Figure: Comparision with the optimal policy.

Index policy is close to optimal.

# Weighted second order throughput cost

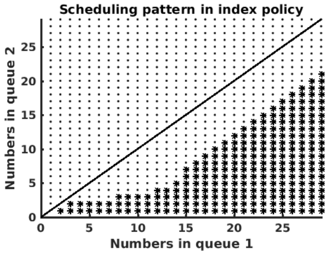


Figure: $q_1 = q_2 = 0.3$ and $d_1 = 4,\ d_2 = 6.$

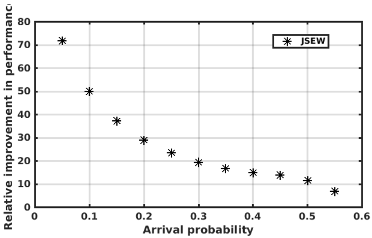


Figure: Relative improvement in performance.

- Scheduling discipline has significant impact for throughput sensitive performance measures.

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# Thank You!!!

# Stationary distribution in Machine Repairman Model

$$\pi_k^{n_k}(m_k) = \frac{P_{m_k}}{\lambda_k(m_k)\left(\sum_{i=0}^{n_k} \frac{P_i}{\lambda_k(i)} + \frac{P_{n_k}}{r_k(n_k+1)}\right)} \ \forall\, m_k = 0, 1, 2, \ldots n_k, \quad (10)$$

$$\pi_k^{n_k}(n_k+1) = \frac{P_{n_k}}{r_k(n_k+1)\left(\sum_{i=0}^{n_k} \frac{P_i}{\lambda_k(i)} + \frac{P_{n_k}}{r_k(n_k+1)}\right)} \quad (11)$$

$$\pi_k^{n_k}(m_k) = 0 \ \forall\, m_k = n_k+2, \ldots \quad (12)$$

Back to Machine Repairman Problem

# Proof of threshold optimality

Define $n^* = \min\{m \in \{0, 1, \cdots\} : S^{\phi^*}(m) = 1\}$

- From the definition of transition rates, all states $m > n^*$ are transient.
- This implies that the optimal average cost is same as the cost under the 0-1 type threshold policy with threshold $n^*$.

Back to Threshold Optimality Result